

# PARI-GP Reference Card

(PARI-GP version 2.3.0)

Note: optional arguments are surrounded by braces {}.

## Starting & Stopping GP

to enter GP, just type its name:

gp

to exit GP, type

\q or quit

## Help

describe function

?function

extended description

??keyword

list of relevant help topics

???pattern

## Input/Output & Defaults

output previous line, the lines before

%, %', %'', etc.

output from line  $n$

%n

separate multiple statements on line

;

extend statement on additional lines

\

extend statements on several lines

{seq<sub>1</sub>; seq<sub>2</sub>};

comment

/\* ... \*/

one-line comment, rest of line ignored

\\ ...

set default  $d$  to  $val$

default({d},{val},flag)

mimic behaviour of GP 1.39

default(compatible,3)

## Metacommands

toggle timer on/off

#

print time for last result

##

print %n in raw format

\a n

print %n in pretty format

\b n

print defaults

\d

set debug level to  $n$

\g n

set memory debug level to  $n$

\gm n

enable/disable logfile

\l {filename}

print %n in pretty matrix format

\m

set output mode (raw, default, prettyprint)

\o n

set  $n$  significant digits

\p n

set  $n$  terms in series

\ps n

quit GP

\q

print the list of PARI types

\t

print the list of user-defined functions

\u

read file into GP

\r filename

write %n to file

\w n filename

## GP Within Emacs

to enter GP from within Emacs:

M-x gp, C-u M-x gp

word completion

(TAB)

help menu window

M-\c

describe function

M-?

display T<sub>E</sub>X'd PARI manual

M-x gpman

set prompt string

M-\p

break line at column 100, insert \

M-\\

PARI metacommand \letter

M-\letter

## Reserved Variable Names

$\pi = 3.14159\dots$

Pi

Euler's constant = .57721...

Euler

square root of -1

I

big-oh notation

O

## PARI Types & Input Formats

t\_INT. Integers

$\pm n$

t\_REAL. Real Numbers

$\pm n.ddd$

t\_INTMOD. Integers modulo  $m$

Mod( $n, m$ )

t\_FRAC. Rational Numbers

$n/m$

t\_COMPLEX. Complex Numbers

$x + y * I$

t\_PADIC.  $p$ -adic Numbers

$x + O(p^k)$

t\_QUAD. Quadratic Numbers

$x + y * \text{quadgen}(D)$

t\_POLMOD. Polynomials modulo  $g$

Mod( $f, g$ )

t\_POL. Polynomials

$a * x^n + \dots + b$

t\_SER. Power Series

$f + O(x^k)$

t\_QFI/t\_QFR. Imag/Real bin. quad. forms

Qfb( $a, b, c, \{d\}$ )

t\_RFRAC. Rational Functions

$f/g$

t\_VEC/t\_COL. Row/Column Vectors

[ $x, y, z$ ], [ $x, y, z$ ]

t\_MAT. Matrices

[ $x, y, z, t; u, v$ ]

t\_LIST. Lists

List([ $x, y, z$ ])

t\_STR. Strings

"aaa"

## Standard Operators

basic operations

+, -, \*, /, ^

i=i+1, i=i-1, i=i\*j, ...

i++, i--, i\*=j,...

euclidean quotient, remainder

$x \backslash y, x \backslash y, x \backslash y, \text{divrem}(x, y)$

shift  $x$  left or right  $n$  bits

$x << n, x >> n$  or  $\text{shift}(x, n)$

comparison operators

<=, <, >=, >, ==, !=

boolean operators (or, and, not)

||, &&, !

sign of  $x = -1, 0, 1$

sign( $x$ )

maximum/minimum of  $x$  and  $y$

max, min( $x, y$ )

integer or real factorial of  $x$

$x!$  or factorial( $x$ )

derivative of  $f$  w.r.t.  $x$

$f'$

## Conversions

### Change Objects

to vector, matrix, set, list, string

Col/Vec, Mat, Set, List, Str

create PARI object ( $x \bmod y$ )

Mod( $x, y$ )

make  $x$  a polynomial of  $v$

Pol( $x, \{v\}$ )

as above, starting with constant term

Polrev( $x, \{v\}$ )

make  $x$  a power series of  $v$

Ser( $x, \{v\}$ )

PARI type of object  $x$

type( $x, \{t\}$ )

object  $x$  with precision  $n$

prec( $x, \{n\}$ )

evaluate  $f$  replacing vars by their value

eval( $f$ )

### Select Pieces of an Object

length of  $x$

# $x$  or length( $x$ )

$n$ -th component of  $x$

component( $x, n$ )

$n$ -th component of vector/list  $x$

$x[n]$

( $m, n$ )-th component of matrix  $x$

$x[m, n]$

row  $m$  or column  $n$  of matrix  $x$

$x[m, ], x[, n]$

numerator of  $x$

numerator( $x$ )

lowest denominator of  $x$

denominator( $x$ )

### Conjugates and Lifts

conjugate of a number  $x$

conj( $x$ )

conjugate vector of algebraic number  $x$

conjvec( $x$ )

norm of  $x$ , product with conjugate

norm( $x$ )

square of  $L^2$  norm of vector  $x$

norml2( $x$ )

lift of  $x$  from Mods

lift, centerlift( $x$ )

## Random Numbers

random integer between 0 and  $N - 1$

random({ $N$ })

get random seed

getrand()

set random seed to  $s$

setrand( $s$ )

## Lists, Sets & Sorting

sort  $x$  by  $k$ th component

vecsort( $x, \{k\}, \{fl = 0\}$ )

Sets (= row vector of strings with strictly increasing entries)

setintersect( $x, y$ )

intersection of sets  $x$  and  $y$

setminus( $x, y$ )

set of elements in  $x$  not belonging to  $y$

setunion( $x, y$ )

union of sets  $x$  and  $y$

setsearch( $x, y, flag$ )

look if  $y$  belongs to the set  $x$

### Lists

create empty list of maximal length  $n$

listcreate( $n$ )

delete all components of list  $l$

listkill( $l$ )

append  $x$  to list  $l$

listput( $l, x, \{i\}$ )

insert  $x$  in list  $l$  at position  $i$

listinsert( $l, x, i$ )

sort the list  $l$

listsort( $l, flag$ )

## Programming & User Functions

Control Statements ( $X$ : formal parameter in expression  $seq$ )

eval.  $seq$  for  $a \leq X \leq b$

for( $X = a, b, seq$ )

eval.  $seq$  for  $X$  dividing  $n$

fordiv( $n, X, seq$ )

eval.  $seq$  for primes  $a \leq X \leq b$

forprime( $X = a, b, seq$ )

eval.  $seq$  for  $a \leq X \leq b$  stepping  $s$

forstep( $X = a, b, s, seq$ )

multivariable for

forvec( $X = v, seq$ )

if  $a \neq 0$ , evaluate  $seq_1$ , else  $seq_2$

if( $a, \{seq_1\}, \{seq_2\}$ )

evaluate  $seq$  until  $a \neq 0$

until( $a, seq$ )

while  $a \neq 0$ , evaluate  $seq$

while( $a, seq$ )

exit  $n$  innermost enclosing loops

break({ $n$ })

start new iteration of  $n$ th enclosing loop

next({ $n$ })

return  $x$  from current subroutine

return( $x$ )

error recovery (try  $seq_1$ )

trap({ $err$ }, { $seq_2$ }, { $seq_1$ })

### Input/Output

prettyprint args with/without newline

printp(), printp1()

print args with/without newline

print(), print1()

read a string from keyboard

input()

reorder priority of variables  $x, y, z$

reorder({ $[x, y, z]$ })

output  $args$  in T<sub>E</sub>X format

printtex( $args$ )

write  $args$  to file

write, write1, writetex( $file, args$ )

read file into GP

read({ $file$ })

### Interface with User and System

allocates a new stack of  $s$  bytes

allocatemem({ $s$ })

execute system command  $a$

system( $a$ )

as above, feed result to GP

extern( $a$ )

install function from library

install( $f, code, \{gpf\}, \{lib\}$ )

alias  $old$  to  $new$

alias( $new, old$ )

new name of function  $f$  in GP 2.0

whatnow( $f$ )

### User Defined Functions

name(formal vars) = local(local vars); seq

struct.member = seq

kill value of variable or function  $x$

kill( $x$ )

declare global variables

global( $x, \dots$ )

## Iterations, Sums & Products

numerical integration

intnum( $X = a, b, expr, flag$ )

sum  $expr$  over divisors of  $n$

sumdiv( $n, X, expr$ )

sum  $X = a$  to  $X = b$ , initialized at  $x$

sum( $X = a, b, expr, \{x\}$ )

sum of series  $expr$

suminf( $X = a, expr$ )

sum of alternating/positive series

sumalt, sumpos

product  $a \leq X \leq b$ , initialized at  $x$

prod( $X = a,$

Vectors & Matrices

dimensions of matrix $x$	<code>matsize(<math>x</math>)</code>
concatenation of $x$ and $y$	<code>concat(<math>x, \{y\}</math>)</code>
extract components of $x$	<code>vecextract(<math>x, y, \{z\}</math>)</code>
transpose of vector or matrix $x$	<code>mattranspose(<math>x</math>)</code> or <code>x-</code>
adjoint of the matrix $x$	<code>matadjoin(<math>x</math>)</code>
eigenvectors of matrix $x$	<code>mateigen(<math>x</math>)</code>
characteristic polynomial of $x$	<code>charpoly(<math>x, \{v\}, flag</math>)</code>
minimal polynomial of $x$	<code>minpoly(<math>x, \{v\}</math>)</code>
trace of matrix $x$	<code>trace(<math>x</math>)</code>

Constructors & Special Matrices

row vec. of $expr$ eval'd at $1 \leq i \leq n$	<code>vector(<math>n, \{i\}, \{expr\}</math>)</code>
col. vec. of $expr$ eval'd at $1 \leq i \leq n$	<code>vectorv(<math>n, \{i\}, \{expr\}</math>)</code>
matrix $1 \leq i \leq m, 1 \leq j \leq n$	<code>matrix(<math>m, n, \{i\}, \{j\}, \{expr\}</math>)</code>
diagonal matrix whose diag. is $x$	<code>matdiagonal(<math>x</math>)</code>
$n \times n$ identity matrix	<code>matid(<math>n</math>)</code>
Hessenberg form of square matrix $x$	<code>mathess(<math>x</math>)</code>
$n \times n$ Hilbert matrix $H_{ij} = (i + j - 1)^{-1}$	<code>mathilbert(<math>n</math>)</code>
$n \times n$ Pascal triangle $P_{ij} = \binom{i}{j}$	<code>matpascal(<math>n - 1</math>)</code>
companion matrix to polynomial $x$	<code>matcompanion(<math>x</math>)</code>

Gaussian elimination

determinant of matrix $x$	<code>matdet(<math>x, flag</math>)</code>
kernel of matrix $x$	<code>matker(<math>x, flag</math>)</code>
intersection of column spaces of $x$ and $y$	<code>matintersect(<math>x, y</math>)</code>
solve $M * X = B$ ( $M$ invertible)	<code>matsolve(<math>M, B</math>)</code>
as solve, modulo $D$ (col. vector)	<code>matsolvemod(<math>M, D, B</math>)</code>
one sol of $M * X = B$	<code>matinverseimage(<math>M, B</math>)</code>
basis for image of matrix $x$	<code>matimage(<math>x</math>)</code>
supplement columns of $x$ to get basis	<code>mat supplement(<math>x</math>)</code>
rows, cols to extract invertible matrix	<code>matindexrank(<math>x</math>)</code>
rank of the matrix $x$	<code>matrank(<math>x</math>)</code>

Lattices & Quadratic Forms

upper triangular Hermite Normal Form	<code>mathnf(<math>x</math>)</code>
HNF of $x$ where $d$ is a multiple of $\det(x)$	<code>mathnfmod(<math>x, d</math>)</code>
elementary divisors of $x$	<code>matsnf(<math>x</math>)</code>
LLL-algorithm applied to columns of $x$	<code>qflll(<math>x, flag</math>)</code>
like <code>qflll</code> , $x$ is Gram matrix of lattice	<code>qflllgram(<math>x, flag</math>)</code>
LLL-reduced basis for kernel of $x$	<code>matkerint(<math>x</math>)</code>
<b>Z</b> -lattice $\longleftrightarrow$ <b>Q</b> -vector space	<code>matrixqz(<math>x, p</math>)</code>
signature of quad form ${}^t y * x * y$	<code>qfsign(<math>x</math>)</code>
decomp into squares of ${}^t y * x * y$	<code>qfgaussred(<math>x</math>)</code>
find up to $m$ sols of ${}^t y * x * y \leq b$	<code>qfminim(<math>x, b, m</math>)</code>
$v, v[i] :=$ number of sols of ${}^t y * x * y = i$	<code>qfrep(<math>x, B, flag</math>)</code>
eigenvals/eigenvecs for real symmetric $x$	<code>qfjacobi(<math>x</math>)</code>

Formal & p-adic Series

truncate power series or $p$ -adic number	<code>truncate(<math>x</math>)</code>
valuation of $x$ at $p$	<code>valuation(<math>x, p</math>)</code>
<b>Dirichlet and Power Series</b>	
Taylor expansion around 0 of $f$ w.r.t. $x$	<code>taylor(<math>f, x</math>)</code>
$\sum a_k b_k t^k$ from $\sum a_k t^k$ and $\sum b_k t^k$	<code>serconvol(<math>x, y</math>)</code>
$f = \sum a_k * t^k$ from $\sum (a_k / k!) * t^k$	<code>serlaplace(<math>f</math>)</code>
reverse power series $F$ so $F(f(x)) = x$	<code>serreverse(<math>f</math>)</code>
Dirichlet series multiplication / division	<code>dirmul, dirdiv(<math>x, y</math>)</code>
Dirichlet Euler product ( $b$ terms)	<code>direuler(<math>p = a, b, expr</math>)</code>

p-adic Functions

Teichmuller character of $x$	<code>teichmuller(<math>x</math>)</code>
Newton polygon of $f$ for prime $p$	<code>newtonpoly(<math>f, p</math>)</code>

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Polynomials & Rational Functions

degree of $f$	<code>poldegree(<math>f</math>)</code>
coefficient of degree $n$ of $f$	<code>polcoeff(<math>f, n</math>)</code>
round coeffs of $f$ to nearest integer	<code>round(<math>f, \{&amp;e\}</math>)</code>
gcd of coefficients of $f$	<code>content(<math>f</math>)</code>
replace $x$ by $y$ in $f$	<code>subst(<math>f, x, y</math>)</code>
discriminant of polynomial $f$	<code>poldisc(<math>f</math>)</code>
resultant of $f$ and $g$	<code>polresultant(<math>f, g, flag</math>)</code>
as above, give $[u, v, d], xu + yv = d$	<code>bezoutres(<math>x, y</math>)</code>
derivative of $f$ w.r.t. $x$	<code>deriv(<math>f, x</math>)</code>
formal integral of $f$ w.r.t. $x$	<code>intformal(<math>f, x</math>)</code>
reciprocal poly $x^{\deg f} f(1/x)$	<code>polrecip(<math>f</math>)</code>
interpol. pol. eval. at $a$	<code>polinterpolate(<math>X, \{Y\}, \{a\}, \{&amp;e\}</math>)</code>
initialize $t$ for Thue equation solver	<code>thueinit(<math>f</math>)</code>
solve Thue equation $f(x, y) = a$	<code>thue(<math>t, a, \{sol\}</math>)</code>

Roots and Factorization

number of real roots of $f, a < x \leq b$	<code>polsturm(<math>f, \{a\}, \{b\}</math>)</code>
complex roots of $f$	<code>polroots(<math>f</math>)</code>
symmetric powers of roots of $f$ up to $n$	<code>polsym(<math>f, n</math>)</code>
roots of $f$ mod $p$	<code>polrootsmod(<math>f, p, flag</math>)</code>
factor $f$	<code>factor(<math>f, \{lim\}</math>)</code>
factorization of $f$ mod $p$	<code>factormod(<math>f, p, flag</math>)</code>
factorization of $f$ over $\mathbb{F}_{p^a}$	<code>factorff(<math>f, p, a</math>)</code>
$p$ -adic fact. of $f$ to prec. $r$	<code>factorpadic(<math>f, p, r, flag</math>)</code>
$p$ -adic roots of $f$ to prec. $r$	<code>polrootspadic(<math>f, p, r</math>)</code>
$p$ -adic root of $f$ cong. to $a$ mod $p$	<code>padicappr(<math>f, a</math>)</code>
Newton polygon of $f$ for prime $p$	<code>newtonpoly(<math>f, p</math>)</code>

Special Polynomials

$n$ th cyclotomic polynomial in var. $v$	<code>polcyclo(<math>n, \{v\}</math>)</code>
$d$ -th degree subfield of $\mathbb{Q}(\zeta_n)$	<code>polsubcyclo(<math>n, d, \{v\}</math>)</code>
$n$ -th Legendre polynomial	<code>pollegendre(<math>n</math>)</code>
$n$ -th Tchebicheff polynomial	<code>pol tchebi(<math>n</math>)</code>
Zagier's polynomial of index $n, m$	<code>polzagier(<math>n, m</math>)</code>

Transcendental Functions

real, imaginary part of $x$	<code>real(<math>x</math>), imag(<math>x</math>)</code>
absolute value, argument of $x$	<code>abs(<math>x</math>), arg(<math>x</math>)</code>
square/ $n$ th root of $x$	<code>sqrtn(<math>x, n, &amp;z</math>)</code>
trig functions	<code>sin, cos, tan, cotan</code>
inverse trig functions	<code>asin, acos, atan</code>
hyperbolic functions	<code>sinh, cosh, tanh</code>
inverse hyperbolic functions	<code>asinh, acosh, atanh</code>
exponential of $x$	<code>exp(<math>x</math>)</code>
natural log of $x$	<code>ln(<math>x</math>)</code> or <code>log(<math>x</math>)</code>
gamma function $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$	<code>gamma(<math>x</math>)</code>
logarithm of gamma function	<code>lngamma(<math>x</math>)</code>
$\psi(x) = \Gamma'(x)/\Gamma(x)$	<code>psi(<math>x</math>)</code>
incomplete gamma function ( $y = \Gamma(s)$ )	<code>incgam(<math>s, x, \{y\}</math>)</code>
exponential integral $\int_x^\infty e^{-t}/t dt$	<code>eint1(<math>x</math>)</code>
error function $2/\sqrt{\pi} \int_x^\infty e^{-t^2} dt$	<code>erfc(<math>x</math>)</code>
dilogarithm of $x$	<code>dilog(<math>x</math>)</code>
$m$ th polylogarithm of $x$	<code>polylog(<math>m, x, flag</math>)</code>
$U$ -confluent hypergeometric function	<code>hyperu(<math>a, b, u</math>)</code>
$J$ -Bessel function $J_{n+1/2}(x)$	<code>besseljh(<math>n, x</math>)</code>
$K$ -Bessel function of index $nu$	<code>besselk(<math>nu, x</math>)</code>

Elementary Arithmetic Functions

vector of binary digits of $ x $	<code>binary(<math>x</math>)</code>
give bit number $n$ of integer $x$	<code>bittest(<math>x, n</math>)</code>
ceiling of $x$	<code>ceil(<math>x</math>)</code>
floor of $x$	<code>floor(<math>x</math>)</code>
fractional part of $x$	<code>frac(<math>x</math>)</code>
round $x$ to nearest integer	<code>round(<math>x, \{&amp;e\}</math>)</code>
truncate $x$	<code>truncate(<math>x, \{&amp;e\}</math>)</code>
gcd/LCM of $x$ and $y$	<code>gcd(<math>x, y</math>), lcm(<math>x, y</math>)</code>
gcd of entries of a vector/matrix	<code>content(<math>x</math>)</code>

Primes and Factorization

add primes in $v$ to the prime table	<code>addprimes(<math>v</math>)</code>
the $n$ th prime	<code>prime(<math>n</math>)</code>
vector of first $n$ primes	<code>primes(<math>n</math>)</code>
smallest prime $\geq x$	<code>nextprime(<math>x</math>)</code>
largest prime $\leq x$	<code>precprime(<math>x</math>)</code>
factorization of $x$	<code>factor(<math>x, \{lim\}</math>)</code>
reconstruct $x$ from its factorization	<code>factorback(<math>fa, \{nf\}</math>)</code>

Divisors

number of distinct prime divisors	<code>omega(<math>x</math>)</code>
number of prime divisors with mult	<code>bigomega(<math>x</math>)</code>
number of divisors of $x$	<code>numdiv(<math>x</math>)</code>
row vector of divisors of $x$	<code>divisors(<math>x</math>)</code>
sum of ( $k$ -th powers of) divisors of $x$	<code>sigma(<math>x, \{k\}</math>)</code>

Special Functions and Numbers

binomial coefficient $\binom{x}{y}$	<code>binomial(<math>x, y</math>)</code>
Bernoulli number $B_n$ as real	<code>bernreal(<math>n</math>)</code>
Bernoulli vector $B_0, B_2, \dots, B_{2n}$	<code>bernvec(<math>n</math>)</code>
$n$ th Fibonacci number	<code>fibonacci(<math>n</math>)</code>
number of partitions of $n$	<code>numbpart(<math>n</math>)</code>
Euler $\phi$ -function	<code>eulerphi(<math>x</math>)</code>
Möbius $\mu$ -function	<code>moebius(<math>x</math>)</code>
Hilbert symbol of $x$ and $y$ (at $p$ )	<code>hilbert(<math>x, y, \{p\}</math>)</code>
Kronecker-Legendre symbol $(\frac{x}{y})$	<code>kronecker(<math>x, y</math>)</code>

Miscellaneous

integer or real factorial of $x$	<code>x!</code> or <code>fact(<math>x</math>)</code>
integer square root of $x$	<code>sqrntint(<math>x</math>)</code>
solve $z \equiv x$ and $z \equiv y$	<code>chinese(<math>x, y</math>)</code>
minimal $u, v$ so $xu + yv = \gcd(x, y)$	<code>bezout(<math>x, y</math>)</code>
multiplicative order of $x$ (intmod) (i=0)	<code>znorder(<math>x, \{o\}</math>)</code>
primitive root mod prime power $q$	<code>znprimroot(<math>q</math>)</code>
structure of $(\mathbb{Z}/n\mathbb{Z})^*$	<code>znstar(<math>n</math>)</code>
continued fraction of $x$	<code>contfrac(<math>x, \{b\}, \{lmax\}</math>)</code>
last convergent of continued fraction $x$	<code>contfracpnqn(<math>x</math>)</code>
best rational approximation to $x$	<code>bestappr(<math>x, k</math>)</code>

True-False Tests

is $x$ the disc. of a quadratic field?	<code>isfundamental(<math>x</math>)</code>
is $x$ a prime?	<code>isprime(<math>x</math>)</code>
is $x$ a strong pseudo-prime?	<code>ispseudoprime(<math>x</math>)</code>
is $x$ square-free?	<code>issquarefree(<math>x</math>)</code>
is $x$ a square?	<code>Zissquare(<math>x, \{&amp;n\}</math>)</code>
is $pol$ irreducible?	<code>polisirreducible(<math>pol</math>)</code>

Based on an earlier version by Joseph H. Silverman  
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Send comments and corrections to <Karim.BELABAS@math.u-psud.fr>

# PARI-GP Reference Card (2)

(PARI-GP version 2.3.0)

## Elliptic Curves

Elliptic curve initially given by 5-tuple  $E = [a_1, a_2, a_3, a_4, a_6]$ . Points are  $[x, y]$ , the origin is  $[0]$ .

Initialize elliptic struct.  $ell$ , i.e create `ellinit( $E, flag$ )`

$a_1, a_2, a_3, a_4, a_6, b_2, b_4, b_6, b_8, c_4, c_6, disc, j$ . This data can be recovered by typing  $ell.a1, \dots, ell.j$ . If  $fl$  omitted, also

$E$  defined over **R**

$x$ -coords. of points of order 2	<code>ell.roots</code>
real and complex periods	<code>ell.omega</code>
associated quasi-periods	<code>ell.eta</code>
volume of complex lattice	<code>ell.area</code>

$E$  defined over  $\mathbf{Q}_p$ ,  $|j|_p > 1$

$x$ -coord. of unit 2 torsion point	<code>ell.roots</code>
Tate's $[u^2, u, q]$	<code>ell.tate</code>
Mestre's $w$	<code>ell.w</code>

change curve  $E$  using  $v = [u, r, s, t]$  `ellchangecurve( $ell, v$ )`

change point  $z$  using  $v = [u, r, s, t]$  `ellchangepoint( $z, v$ )`

cond, min mod, Tamagawa num  $[N, v, c]$  `ellglobalred( $ell$ )`

Kodaira type of  $p$  fiber of  $E$  `ellllocalred( $ell, p$ )`

add points  $z_1 + z_2$  `elladd( $ell, z_1, z_2$ )`

subtract points  $z_1 - z_2$  `ellsub( $ell, z_1, z_2$ )`

compute  $n \cdot z$  `ellpow( $ell, z, n$ )`

check if  $z$  is on  $E$  `ellisoncurve( $ell, z$ )`

order of torsion point  $z$  `ellorder( $ell, z$ )`

torsion subgroup with generators `elltors( $ell$ )`

$y$ -coordinates of point(s) for  $x$  `ellordinate( $ell, x$ )`

canonical bilinear form taken at  $z_1, z_2$  `ellbil( $ell, z_1, z_2$ )`

canonical height of  $z$  `ellheight( $ell, z, flag$ )`

height regulator matrix for pts in  $x$  `ellheightmatrix( $ell, x$ )`

$p$ th coeff  $a_p$  of  $L$ -function,  $p$  prime `ellap( $ell, p$ )`

$k$ th coeff  $a_k$  of  $L$ -function `ellak( $ell, k$ )`

vector of first  $n$   $a_k$ 's in  $L$ -function `ellan( $ell, n$ )`

$L(E, s)$ , set  $A \approx 1$  `elllseries( $ell, s, \{A\}$ )`

root number for  $L(E, \cdot)$  at  $p$  `ellrootno( $ell, \{p\}$ )`

modular parametrization of  $E$  `elltaniyama( $ell$ )`

point  $[\wp(z), \wp'(z)]$  corresp. to  $z$  `ellztopoint( $ell, z$ )`

complex  $z$  such that  $p = [\wp(z), \wp'(z)]$  `ellpointtoz( $ell, p$ )`

## Elliptic & Modular Functions

arithmetic-geometric mean `agm( $x, y$ )`

elliptic  $j$ -function  $1/q + 744 + \dots$  `ellj( $x$ )`

Weierstrass  $\sigma$  function `ellsigma( $ell, z, flag$ )`

Weierstrass  $\wp$  function `ellwp( $ell, \{z\}, flag$ )`

Weierstrass  $\zeta$  function `ellzeta( $ell, z$ )`

modified Dedekind  $\eta$  func.  $\prod(1 - q^n)$  `eta( $x, flag$ )`

Jacobi sine theta function `theta( $q, z$ )`

k-th derivative at  $z=0$  of `thetanullk( $q, k$ )`

Weber's  $f$  functions `weber( $x, flag$ )`

Riemann's zeta  $\zeta(s) = \sum n^{-s}$  `zeta( $s$ )`

## Graphic Functions

crude graph of  $expr$  between  $a$  and  $b$  `plot( $X = a, b, expr$ )`

**High-resolution plot** (immediate plot)

plot  $expr$  between  $a$  and  $b$  `plotoh( $X = a, b, expr, flag, \{n\}$ )`

plot points given by lists  $lx, ly$  `plotdraw( $lx, ly, flag$ )`

terminal dimensions `plotsizes()`

### Rectwindow functions

init window  $w$ , with size  $x, y$  `plotinit( $w, x, y$ )`

erase window  $w$  `plotkill( $w$ )`

copy  $w$  to  $w_2$  with offset  $(dx, dy)$  `plotcopy( $w, w_2, dx, dy$ )`

scale coordinates in  $w$  `plotscale( $w, x_1, x_2, y_1, y_2$ )`

`plotoh` in  $w$  `plotrecth( $w, X = a, b, expr, flag, \{n\}$ )`

`plotdraw` in  $w$  `plotrecthdraw( $w, data, flag$ )`

draw window  $w_1$  at  $(x_1, y_1), \dots$  `plotdraw( $[[w_1, x_1, y_1], \dots]$ )`

### Low-level Rectwindow Functions

set current drawing color in  $w$  to  $c$  `plotcolor( $w, c$ )`

current position of cursor in  $w$  `plotcursor( $w$ )`

write  $s$  at cursor's position `plotstring( $w, s$ )`

move cursor to  $(x, y)$  `plotmove( $w, x, y$ )`

move cursor to  $(x + dx, y + dy)$  `plotrmove( $w, dx, dy$ )`

draw a box to  $(x_2, y_2)$  `plotbox( $w, x_2, y_2$ )`

draw a box to  $(x + dx, y + dy)$  `plotrbox( $w, dx, dy$ )`

draw polygon `plotlines( $w, lx, ly, flag$ )`

draw points `plotpoints( $w, lx, ly$ )`

draw line to  $(x + dx, y + dy)$  `plotrline( $w, dx, dy$ )`

draw point  $(x + dx, y + dy)$  `plotrpoint( $w, dx, dy$ )`

### Postscript Functions

as `plotoh` `psplotoh( $X = a, b, expr, flag, \{n\}$ )`

as `plotdraw` `psplotdraw( $lx, ly, flag$ )`

as `plotdraw` `psdraw( $[[w_1, x_1, y_1], \dots]$ )`

## Binary Quadratic Forms

create  $ax^2 + bxy + cy^2$  (distance  $d$ ) `qfb( $a, b, c, \{d\}$ )`

reduce  $x$  ( $s = \sqrt{D}$ ,  $l = \lfloor s \rfloor$ ) `qfbred( $x, flag, \{D\}, \{l\}, \{s\}$ )`

composition of forms  $x*y$  or `qfbnucomp( $x, y, l$ )`

$n$ -th power of form  $x^n$  or `qfbnupow( $x, n$ )`

composition without reduction `qfbcompraw( $x, y$ )`

$n$ -th power without reduction `qfbpowraw( $x, n$ )`

prime form of disc.  $x$  above prime  $p$  `qfbprimeform( $x, p$ )`

class number of disc.  $x$  `qfbclassno( $x$ )`

Hurwitz class number of disc.  $x$  `qfbhclassno( $x$ )`

## Quadratic Fields

quadratic number  $\omega = \sqrt{x}$  or  $(1 + \sqrt{x})/2$  `quadgen( $x$ )`

minimal polynomial of  $\omega$  `quadpoly( $x$ )`

discriminant of  $\mathbf{Q}(\sqrt{D})$  `quaddisc( $x$ )`

regulator of real quadratic field `quadregulator( $x$ )`

fundamental unit in real  $\mathbf{Q}(x)$  `quadunit( $x$ )`

class group of  $\mathbf{Q}(\sqrt{D})$  `quadclassunit( $D, flag, \{t\}$ )`

Hilbert class field of  $\mathbf{Q}(\sqrt{D})$  `quadhilbert( $D, flag$ )`

ray class field modulo  $f$  of  $\mathbf{Q}(\sqrt{D})$  `quadray( $D, f, flag$ )`

## General Number Fields: Initializations

A number field  $K$  is given by a monic irreducible  $f \in \mathbf{Z}[X]$ .

init number field structure  $nf$  `nfinit( $f, flag$ )`

### nf members:

polynomial defining $nf$ , $f(\theta) = 0$	<code>nf.pol</code>
number of real/complex places	<code>nf.r1, nf.r2</code>
discriminant of $nf$	<code>nf.disc</code>
$T_2$ matrix	<code>nf.t2</code>
vector of roots of $f$	<code>nf.roots</code>
integral basis of $\mathbf{Z}_K$ as powers of $\theta$	<code>nf.zk</code>
different	<code>nf.diff</code>
codifferent	<code>nf.codiff</code>

recompute  $nf$  using current precision `nfnewprec( $nf$ )`

init relative  $rmf$  given by  $g = 0$  over  $K$  `rmfinit( $nf, g$ )`

init  $bnf$  structure `bnfinit( $f, flag$ )`

**bnf members:** same as  $nf$ , plus

underlying $nf$	<code>bnf.nf</code>
classgroup	<code>bnf.clgp</code>
regulator	<code>bnf.reg</code>
fundamental units	<code>bnf.fu</code>
torsion units	<code>bnf.tu</code>
$[tu, fu]$	<code>bnf.tufu</code>

compute a  $bnf$  from small  $bnf$  `bnfmake( $sbnf$ )`

add  $S$ -class group and units, yield  $bnf$  s `bnfsunit( $nf, S$ )`

init class field structure  $bnr$  `bnrinit( $bnf, m, flag$ )`

**bnr members:** same as  $bnf$ , plus

underlying $bnf$	<code>bnr.bnf</code>
structure of $(\mathbf{Z}_K/m)^*$	<code>bnr.zkst</code>

## Simple Arithmetic Invariants (nf)

Elements are rational numbers, polynomials, polmods, or column vectors (on integral basis  $nf.zk$ ).

integral basis of field def. by  $f = 0$       **nfbasis**( $f$ )  
field discriminant of field  $f = 0$       **nfdisc**( $f$ )  
reverse polmod  $a = A(X) \bmod T(X)$       **modreverse**( $a$ )  
Galois group of field  $f = 0$ ,  $\deg f \leq 11$       **polgalois**( $f$ )  
smallest poly defining  $f = 0$       **polredabs**( $f, flag$ )  
small polys defining subfields of  $f = 0$       **polred**( $f, flag, \{p\}$ )  
small polys defining suborders of  $f = 0$       **polredord**( $f$ )  
poly of degree  $\leq k$  with root  $x \in \mathbf{C}$       **algdep**( $x, k$ )  
small linear rel. on coords of vector  $x$       **lindep**( $x$ )  
are fields  $f = 0$  and  $g = 0$  isomorphic?      **nfisism**( $f, g$ )  
is field  $f = 0$  a subfield of  $g = 0$ ?      **nfisincl**( $f, g$ )  
compositum of  $f = 0$ ,  $g = 0$       **polcompositum**( $f, g, flag$ )  
basic element operations (prefix **nfelt**):

(**nfelt**)**mul**, **pow**, **div**, **diveuc**, **mod**, **divrem**, **val**  
express  $x$  on integer basis      **nfalgtobasis**( $nf, x$ )  
express element  $x$  as a polmod      **nfbasistoalg**( $nf, x$ )  
quadratic Hilbert symbol (at  $p$ )      **nfhilbert**( $nf, a, b, \{p\}$ )  
roots of  $g$  belonging to  $nf$       **nfroots**( $\{nf\}, g$ )  
factor  $g$  in  $nf$       **nfactor**( $nf, g$ )  
factor  $g$  mod prime  $pr$  in  $nf$       **nfactormod**( $nf, g, pr$ )  
number of roots of unity in  $nf$       **nfrootsof1**( $nf$ )  
conjugates of a root  $\theta$  of  $nf$       **nfgaloisconj**( $nf, flag$ )  
apply Galois automorphism  $s$  to  $x$       **nfgaloisapply**( $nf, s, x$ )  
subfields (of degree  $d$ ) of  $nf$       **nfsubfields**( $nf, \{d\}$ )

### Dedekind Zeta Function $\zeta_K$

$\zeta_K$  as Dirichlet series,  $N(I) < b$       **dirzetak**( $nf, b$ )  
init  $nfz$  for field  $f = 0$       **zetakinit**( $f$ )  
compute  $\zeta_K(s)$       **zetak**( $nfz, s, flag$ )  
Artin root number of  $K$       **bnrrootnumber**( $bnr, chi, flag$ )

## Class Groups & Units (bnf, bnr)

$a_1, \{a_2\}, \{a_3\}$  usually  $bnr, subgp$  or  $bnf, module, \{subgp\}$   
remove GRH assumption from  $bnf$       **bnfcertify**( $bnf$ )  
expo. of ideal  $x$  on class gp      **bnfisprincipal**( $bnf, x, flag$ )  
expo. of ideal  $x$  on ray class gp      **bnrisprincipal**( $bnr, x, flag$ )  
expo. of  $x$  on fund. units      **bnfisunit**( $bnf, x$ )  
as above for  $S$ -units      **bnfissunit**( $bnfs, x$ )  
fundamental units of  $bnf$       **bnfunit**( $bnf$ )  
signs of real embeddings of  $bnf.fu$       **bnfsignunit**( $bnf$ )

### Class Field Theory

ray class group structure for mod.  $m$       **bnrclass**( $bnf, m, flag$ )  
ray class number for mod.  $m$       **bnrclassno**( $bnf, m$ )  
discriminant of class field ext      **bnrdisc**( $a_1, \{a_2\}, \{a_3\}$ )  
ray class numbers,  $l$  list of mods      **bnrclassnolist**( $bnf, l$ )  
discriminants of class fields      **bnrdisclist**( $bnf, l, \{arch\}, flag$ )  
decode output from **bnrdisclist**      **bnfdecodemodule**( $nf, fa$ )  
is modulus the conductor?      **bnrisconductor**( $a_1, \{a_2\}, \{a_3\}$ )  
conductor of character  $chi$       **bnrconductorofchar**( $bnr, chi$ )  
conductor of extension      **bnrconductor**( $a_1, \{a_2\}, \{a_3\}, flag$ )  
conductor of extension def. by  $g$       **rnfconductor**( $bnf, g$ )  
Artin group of ext. def'd by  $g$       **rnfnormgroup**( $bnr, g$ )  
subgroups of  $bnr$ , index  $\leq b$       **subgrouplist**( $bnr, b, flag$ )  
rel. eq. for class field def'd by  $sub$       **rnfkummer**( $bnr, sub, \{d\}$ )  
same, using Stark units (real field)      **bnrstark**( $bnr, sub, flag$ )

## PARI-GP Reference Card (2)

(PARI-GP version 2.3.0)

### Ideals

Ideals are elements, primes, or matrix of generators in HNF.  
is  $id$  an ideal in  $nf$ ?      **nfisideal**( $nf, id$ )  
is  $x$  principal in  $bnf$ ?      **bnfisprincipal**( $bnf, x$ )  
principal ideal generated by  $x$       **idealprincipal**( $nf, x$ )  
principal idele generated by  $x$       **ideleprincipal**( $nf, x$ )  
give  $[a, b]$ , s.t.  $a\mathbf{Z}_K + b\mathbf{Z}_K = x$       **idealtwoelt**( $nf, x, \{a\}$ )  
put ideal  $a$  ( $a\mathbf{Z}_K + b\mathbf{Z}_K$ ) in HNF form      **idealhnf**( $nf, a, \{b\}$ )  
norm of ideal  $x$       **idealnrm**( $nf, x$ )  
minimum of ideal  $x$  (direction  $v$ )      **idealmin**( $nf, x, v$ )  
LLL-reduce the ideal  $x$  (direction  $v$ )      **idealred**( $nf, x, \{v\}$ )

### Ideal Operations

add ideals  $x$  and  $y$       **idealadd**( $nf, x, y$ )  
multiply ideals  $x$  and  $y$       **idealmul**( $nf, x, y, flag$ )  
intersection of ideals  $x$  and  $y$       **idealintersect**( $nf, x, y, flag$ )  
 $n$ -th power of ideal  $x$       **idealpow**( $nf, x, n, flag$ )  
inverse of ideal  $x$       **idealinv**( $nf, x$ )  
divide ideal  $x$  by  $y$       **idealdiv**( $nf, x, y, flag$ )  
Find  $(a, b) \in x \times y$ ,  $a + b = 1$       **idealaddtoone**( $nf, x, \{y\}$ )

### Primes and Multiplicative Structure

factor ideal  $x$  in  $nf$       **idealfactor**( $nf, x$ )  
recover  $x$  from its factorization in  $nf$       **factorback**( $x, nf$ )  
decomposition of prime  $p$  in  $nf$       **idealprimedec**( $nf, p$ )  
valuation of  $x$  at prime ideal  $pr$       **idealval**( $nf, x, pr$ )  
weak approximation theorem in  $nf$       **idealchinese**( $nf, x, y$ )  
give  $bid$  = structure of  $(\mathbf{Z}_K/id)^*$       **idealstar**( $nf, id, flag$ )  
discrete log of  $x$  in  $(\mathbf{Z}_K/bid)^*$       **ideallog**( $nf, x, bid$ )  
**idealstar** of all ideals of norm  $\leq b$       **ideallist**( $nf, b, flag$ )  
add archimedean places      **ideallistarch**( $nf, b, \{ar\}, flag$ )  
init **prmod** structure      **nfmodprinit**( $nf, pr$ )  
kernel of matrix  $M$  in  $(\mathbf{Z}_K/pr)^*$       **nfkermodpr**( $nf, M, prmod$ )  
solve  $Mx = B$  in  $(\mathbf{Z}_K/pr)^*$       **nfsolvemodpr**( $nf, M, B, prmod$ )

## Galois theory over $q$

initializes a Galois group structure      **galoisinit**( $pol, \{den\}$ )  
action of  $p$  in **nfgaloisconj** form      **galoispermopol**( $G, \{p\}$ )  
identifies as abstract group      **galoisidentify**( $G$ )  
exports a group for GAP or MAGMA      **galoisexport**( $G, flag$ )  
subgroups of the Galois group  $G$       **galoissubgroups**( $G$ )  
subfields from subgroups of  $G$       **galoissubfields**( $G, flag, \{v\}$ )  
fixed field      **galoisfixedfield**( $G, perm, flag, \{v\}$ )  
is  $G$  abelian?      **galoisisabelian**( $G, flag$ )  
abelian number fields      **galoissubcyclo**( $N, H, flag, \{v\}$ )

## Relative Number Fields (rnf)

Extension  $L/K$  is defined by  $g \in K[x]$ . We have  $order \subset L$ .  
absolute equation of  $L$       **rnfequation**( $nf, g, flag$ )  
relative **nfalgtobasis**      **rnfalgtobasis**( $rnf, x$ )  
relative **nfbasistoalg**      **rnfbasistoalg**( $rnf, x$ )  
relative **idealhnf**      **rnfidealhnf**( $rnf, x$ )  
relative **idealmul**      **rnfidealmul**( $rnf, x, y$ )  
relative **idealtwoelt**      **rnfidealtwoelt**( $rnf, x$ )

### Lifts and Push-downs

absolute  $\rightarrow$  relative repres. for  $x$       **rnfeltabstorel**( $rnf, x$ )  
relative  $\rightarrow$  absolute repres. for  $x$       **rnfeltreltoabs**( $rnf, x$ )  
lift  $x$  to the relative field      **rnfeltup**( $rnf, x$ )  
push  $x$  down to the base field      **rnfeltdown**( $rnf, x$ )  
idem for  $x$  ideal: (**rnfideal**)**reltoabs**, **abstorel**, **up**, **down**

### Projective $\mathbf{Z}_K$ -modules, maximal order

relative **polred**      **rnfpolred**( $nf, g$ )  
relative **polredabs**      **rnfpolredabs**( $nf, g$ )  
characteristic poly. of  $a \bmod g$       **rnfcharpoly**( $nf, g, a, \{v\}$ )  
relative Dedekind criterion, prime  $pr$       **rnfdedekind**( $nf, g, pr$ )  
discriminant of relative extension      **rnfdisc**( $nf, g$ )  
pseudo-basis of  $\mathbf{Z}_L$       **rnfpseudobasis**( $nf, g$ )  
relative HNF basis of  $order$       **rnfhnfbasis**( $bnf, order$ )  
reduced basis for  $order$       **rnflllgram**( $nf, g, order$ )  
determinant of pseudo-matrix  $A$       **rnfdet**( $nf, A$ )  
Steinitz class of  $order$       **rnfsteynitz**( $nf, order$ )  
is  $order$  a free  $\mathbf{Z}_K$ -module?      **rnfisfree**( $bnf, order$ )  
true basis of  $order$ , if it is free      **rnfbasis**( $bnf, order$ )

### Norms

absolute norm of ideal  $x$       **rnfidealnrmabs**( $rnf, x$ )  
relative norm of ideal  $x$       **rnfidealnrmrel**( $rnf, x$ )  
solutions of  $N_{K/\mathbf{Q}}(y) = x \in \mathbf{Z}$       **bnfisintnorm**( $bnf, x$ )  
is  $x \in \mathbf{Q}$  a norm from  $K$ ?      **bnfisnorm**( $bnf, x, flag$ )  
initialize  $T$  for norm eq. solver      **rnfisnorminit**( $K, pol, flag$ )  
is  $a \in K$  a norm from  $L$ ?      **rnfisnorm**( $T, a, flag$ )

Based on an earlier version by Joseph H. Silverman

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